MATH 590: QUIZ 2

Throughout V will denote a vector space over F, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

1. Define what it means for the vectors $v_1, \dots, v_r$ in V to be linearly dependent. (2 points)

Solution. The vectors are linearly dependent if there exists a linear combination $\alpha_1 v_1 + \cdots + \alpha_r v_r = \vec{0}$, with each $\alpha_i \in F$ and at least one $\alpha_i \neq 0$.

2. State an equivalent condition for $v_1, \dots, v_r \in V$ to be linearly dependent (as in the proposition from class). (2 points)

Solution. The vectors $v_1, \dots, v_r$ are linearly dependent if and only if some $v_i \in \text{Span}\{v_1, \dots, \hat{i}, \dots, v_r\}$.

3. State a condition for $v_1, \dots, v_r$ to be linearly independent, other than saying the vectors are not linearly dependent. (2 points)

Solution. The vectors $v_1, \dots, v_r$ are linearly independent if and only if whenever $\alpha_1 v_1 + \cdots + \alpha_r v_r = \vec{0}$, then $\alpha_i = 0$, for all i.

4. Show that any three non-zero vectors in $\mathbb{R}^2$ are linearly dependent.

Solution. Suppose $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}, v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$ and $v_2 = \begin{pmatrix} e \\ f \end{pmatrix}$. Here are two possible (among many) solutions.

First solution: Let $A = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$. Then v_1, v_2, v_3 are linearly dependent, if the homogeneous systems of equations $A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$ has a non-trivial solution. This is a homogeneous system of two equations in three unknowns, which (from Math 290) has infinitely many solutions, and thus a non-trivial solution.

Variation of first solution: Starting with the same homogeneous system of equations, we have the augmeted matrix

$$\begin{pmatrix} a & c & e & | & 0 \\ b & d & f & | & 0 \end{pmatrix}$$

which reduces via elementary row operations to a matrix of one of the following types:

$$\begin{pmatrix} 1 & * & * & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * & | & 0 \\ 0 & 1 & * & | & 0 \end{pmatrix}, \begin{pmatrix} 1 & * & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

which all yield infinitely many solutions, and thus a non-trivial solution.

Second solution: If some pair of the vectors are already linearly dependent, then we are done. Thus, suppose any two of the given vectors are linearly independent. In particular, v_1 and v_2 are linearly independent. It suffices to show that we can find $x, y \in F$ such that $v_3 = xv_1 + yv_3$. If we set $B := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, this means we must solve the system of equations given by $B \cdot \begin{pmatrix} x \\ y \end{pmatrix} = v_3$. The columns of B are linearly independent, and by Math 290, $\det(B) \neq 0$. Thus, B is invertible. We may therefore solve the system of equations $B \cdot \begin{pmatrix} x \\ y \end{pmatrix} = v_3$, by writing $\begin{pmatrix} x \\ y \end{pmatrix} = B^{-1} v_3$.